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Variational and numerical approach to a steady-state rolling problem

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Abstract A steady-state rolling problem with rigid–plastic, incompressible material and nonlocal Coulomb-contact friction condition is considered. The corresponding primal, penalty and regularized penalty variational formulations are presented and studied. It is shown that the solutions of the penalty and regularized penalty variational problems converge to the solutions of the primal and penalty variational problems, when the penalty and regularization parameters tend to zero. The finite-element approximation of the regularized penalty problem is presented and analysed. An algorithm, combining the finite-element method with convergent successive iterations method of secant-modulus is proposed and applied to solve an illustrative example.

Keywords FE analysis · Nonlocal friction · Rigid–plastic material · Variational formulations · Weak solutions

1 Introduction

Following the thorough mechanical and computational study of continuous metal-forming processes, such as extrusion, drawing and rolling [1], [2, pp. 1–43], [3], it was recently found that, within the framework of the flow theory of plasticity, these problems ([4–6]) can be mathematically formulated and analysed analogously to the frictional-contact problems in elasticity [7, Chap. 3], [8, Chap. 13], [9, Chap. 5], [10, Chaps. 7, 10, 11], [11, Part 2], [12, Sects. 2, 3]. These approaches use and extend the ideas and methods that were developed for contact problems and their variational and numerical investigation.

The aim of this work is to state and study the solution of a boundary-value problem describing an isothermal, steady-state rolling process with rigid–plastic, incompressible material and non-local Coulomb-friction contact conditions. The corresponding primal and penalty variational problems are derived and the properties of the constituted functionals are studied. Existence and uniqueness results are briefly commented on. Combining appropriately the penalty parameters and taking the limit to zero, we prove the convergence of the solution of the penalty problem to the solution of the primal problem. After a regularization of the nondifferentiable functionals of the penalty problem, combining the regulazation parameters and taking the limit to zero, we demonstrate the convergence of the solution of the regularized problem to the solution of the nonregularized one. A finite-element approximation of the regularized penalty problem is presented and an a priori error estimate is obtained. A convergent algorithm,

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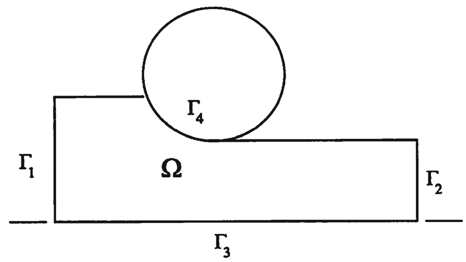

Fig. 1 The setting of steady-state rolling

combining the finite-element method and the iterative method of secant-modulus type is proposed and applied to solve an example problem. The numerical results are obtained and discussed.

2 Statement of the problem

We consider an isotropic, rigid–plastic and incompressible metallic body (workpiece) occupying the domain $\Omega \subset \mathbb{R}^k$, $k = 2, 3$, (Fig. 1), at isothermal and steady-state conditions. The boundary of the domain consists of four open disjoint subsets $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, as $\Gamma_1 \cup \Gamma_2$ and is assumed traction-free; Γ_3 is the boundary of symmetry and Γ_4 is the contact boundary. Here we identify a point of $\bar{\Omega} = \Omega \cup \Gamma$ by its Cartesian coordinates $\mathbf{x} = \{x_i\}$, $(1 \le i \le k)$, and use the standard index notations. We assume that the workpiece material satisfies the following yield criterion and flow rule:

$$F(\sigma_{ij}, \dot{\varepsilon}_{ij}) \equiv \bar{\sigma}^2 - \sigma_p^2 = 0, \quad \dot{e}_{ij} = \frac{3}{2}\frac{\dot{\bar{\varepsilon}}}{\bar{\sigma}} s_{ij}. \tag{2.1}$$

The equivalent stress and strain-rate are given by the expressions

$$\bar{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}}, \quad \dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3} \dot{e}_{ij} \dot{e}_{ij}}, \tag{2.2}$$

where $\sigma_H = \frac{1}{3}\sigma_{ii}$, $\dot{\varepsilon}_V = \dot{\varepsilon}_{ii}$, are the hydrostatic pressure and the volume dilatation strain-rate; $s_{ij} = \sigma_{ij} - \sigma_H \delta_{ij}$, $\dot{e}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\dot{\varepsilon}_V \delta_{ij}$, are the components of the deviatoric stress and the strain-rate tensors, σ_{ij} and $\dot{\varepsilon}_{ij}$, $1 \le i, j \le k$, are the components of the stress and the strain-rate tensors. We further assume that the uniaxial yield limit σ_p depends on the equivalent strain-rate $\dot{\bar{\varepsilon}}$ and that the following decomposition holds:

$$\sigma_p(\dot{\bar{\varepsilon}}) = \sigma_{p0} + \sigma_{p1}(\dot{\bar{\varepsilon}}), \tag{2.3}$$

where $\sigma_{p0} = \sigma_p(0) \ge 0$ is the initial yield limit of the material and $\sigma_{p1}(\dot{\bar{\varepsilon}})$, with $\sigma_{p1}(0) = 0$, is assumed to be a monotonically increasing, almost everywhere differentiable function such that:

$$\eta_2 \le \sigma'_{p1}(\dot{\bar{\varepsilon}}) \le \frac{\sigma_{p1}(\dot{\bar{\varepsilon}})}{\dot{\bar{\varepsilon}}} \le \eta_1, \quad \forall \dot{\bar{\varepsilon}} \in [0, \infty), \tag{2.4}$$

where η_1 and η_2 are positive constants and where a prime denotes differentiation with respect to the argument. We state the following boundary-value problem:

Problem 1 Find the velocity $\mathbf{u} = \{u_i\}$ and the stress $\boldsymbol{\sigma} = \{\sigma_{ij}\}$ fields, satisfying the following equations and relations:

– equation of equilibrium

$$\sigma_{ij,j} = 0 \quad \text{in } \Omega, \tag{2.5}$$

– incompressibility condition

$$\dot{\varepsilon}_V = 0 \quad \text{in } \Omega, \tag{2.6}$$

– constitutive equations

$$\sigma_{ij} = \frac{2}{3}\frac{\sigma_p}{\dot{\bar{\varepsilon}}}\dot{\varepsilon}_{ij} + \sigma_H \delta_{ij}, \tag{2.7}$$

– strain-rate velocity relations

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.8}$$

– boundary conditions

$$\sigma_{ij} n_j = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \tag{2.9}$$

$$\boldsymbol{\sigma}_T = \mathbf{0}, \quad u_N = 0 \quad \text{on } \Gamma_3, \tag{2.10}$$

$$\begin{aligned} &u_N = 0, \quad \text{and} \\ &\text{if} \quad |\boldsymbol{\sigma}_T(\mathbf{u})| < \tau_f(\mathbf{u}), \quad \text{then } \mathbf{u}_T - \mathbf{u}_{TR} = \mathbf{0}, \\ &\text{if} \quad |\boldsymbol{\sigma}_T(\mathbf{u})| = \tau_f(\mathbf{u}), \quad \text{then } \exists \text{ const. } \lambda \geq 0, \\ &\text{such that} \quad \mathbf{u}_T - \mathbf{u}_{TR} = -\lambda \boldsymbol{\sigma}_T(\mathbf{u}) \text{ on } \Gamma_4. \end{aligned} \tag{2.11}$$

Here δ_{ij} is the Kronecker symbol; $\mathbf{n} = \{n_i\}$ is the outward unit normal vector with respect to Γ; u_N, $\mathbf{u}_T$ and σ_N, $\boldsymbol{\sigma}_T$ are the normal and tangential components of the velocity and the stress vector; $\mathbf{u}_{TR}$ is the tangential component of the roll velocity; $\tau_f(\mathbf{u})$ is the shear-strength limit for the material of the roll-workpiece interface, expressed by the nonlocal Coulomb-friction law:

$$\tau_f(\mathbf{u}) = \mu_f(\mathbf{x})\bar{\sigma}_N(\mathbf{u}), \tag{2.12}$$

$$\bar{\sigma}_N(\mathbf{u}(\mathbf{x})) = \frac{1}{|\Gamma_h|}\int_{\Gamma_h} w_h(\mathbf{x}-\mathbf{y})(-\sigma_N(\mathbf{u}(\mathbf{y})))\mathrm{d}y, \quad \mathbf{x} \in \Gamma_4, \tag{2.13}$$

$$w_h(\mathbf{x}-\mathbf{y}) = \begin{cases} 1 & \text{if} \quad |\mathbf{x}-\mathbf{y}| < h, \\ 0 & \text{if} \quad |\mathbf{x}-\mathbf{y}| \geq h, \end{cases} \tag{2.14}$$

where $\mu_f(\mathbf{x})$ is the friction coefficient, $\bar{\sigma}_N(\mathbf{u}) \geq 0$ is the mollified normal stress on Γ_4 ([5], [10, Chap. 11]) and $w_h(\mathbf{x}-\mathbf{y})$ is the mollification kernel.

3 Variational formulations

Let us denote by $\mathbf{V}$ and $\mathbf{H}$ the following Hilbert spaces

$$\mathbf{V} = \{\mathbf{v} : \mathbf{v} \in (H^1(\Omega))^k,\ v_N = 0 \text{ on } \Gamma_3\}, \quad \mathbf{H} = (H^0(\Omega))^k \equiv (L_2(\Omega))^k, \quad \mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}',$$

where $\mathbf{V}'$ and $\mathbf{H}'$ are their dual spaces. By $(H^m(\Omega))^k$, with m a nonnegative integer, we denote the Hilbert space of vector-valued functions defined in Ω

$$(H^m(\Omega))^k = \big\{\mathbf{v} = \{v_i\} : D^\alpha v_i \in L_2(\Omega), 1 \leq i \leq k, 0 \leq |\boldsymbol{\alpha}| \leq m\big\},$$

with the inner product and norm

$$(\mathbf{u}, \mathbf{v})_m = \int_\Omega \sum_{|\alpha|=0}^{m} \left(\sum_{i=1}^{k} D^\alpha u_i D^\alpha v_i\right)\mathrm{d}x,$$

$$\|\mathbf{u}\|_m = (\mathbf{u}, \mathbf{u})_m^{1/2} = \left(\int_\Omega \sum_{|\alpha|=0}^{m} \left(\sum_{i=1}^{k} |D^\alpha u_i|^2\right)\mathrm{d}x\right)^{1/2},$$

where

$$D^\alpha v_i = \frac{\partial^{|\alpha|} v_i}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_k^{\alpha_k}},$$

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_k) \in \mathbb{Z}^k, \quad \alpha_i \geq 0, \quad |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

We further define on $\mathbf{V}$ the following inner product and norm

$$(\mathbf{u},\mathbf{v})_V=\int_\Omega \dot\varepsilon_{ij}(\mathbf{u})\dot\varepsilon_{ij}(\mathbf{v})\mathrm{d}\mathbf{x}+\int_{\Gamma_4} u_N v_N \mathrm{d}\Gamma,\quad \|\mathbf{u}\|_V=(\mathbf{u},\mathbf{u})_V^{1/2},\quad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}, \tag{3.1}$$

and denote by $\mathbf{W}$ the following subspace of $\mathbf{V}$

$$\mathbf{W}=\left\{\mathbf{v}:\mathbf{v}\in\mathbf{V},\quad v_{i,i}=0\quad \text{in }\Omega,\quad v_N=0\quad \text{on }\Gamma_4\right\}.$$

We shall also use the space $H^{1/2}(\Gamma_4)\subset L_2(\Gamma_4)$ of traces $v_N=\boldsymbol{\gamma}_0(\mathbf{v})\cdot\mathbf{n}$ on Γ_4 of all $\mathbf{v}\in\mathbf{V}$, with the norm

$$\|v_N\|_{1/2,\Gamma}=\inf_{\mathbf{v}\in\mathbf{V}}\left\{\|\mathbf{v}\|_V: v_N=\boldsymbol{\gamma}_0(\mathbf{v})\cdot\mathbf{n}\right\},$$

where $\boldsymbol{\gamma}_0:(H^1(\Omega))^k\to(H^{1/2}(\Gamma))^k$ is the trace operator.

Then, multiplying (2.5) by $(\mathbf{v}-\mathbf{u})\in\mathbf{W}$, in the inner product sense, applying Green's formula and taking into account the boundary conditions, we obtain

$$\int_\Omega \sigma_{ij}(\mathbf{u})(\dot\varepsilon_{ij}(\mathbf{v})-\dot\varepsilon_{ij}(\mathbf{u}))\mathrm{d}x+\int_{\Gamma_4}\tau_f(\mathbf{u})|\mathbf{v}_T-\mathbf{u}_{TR}|\mathrm{d}\Gamma-\int_{\Gamma_4}\tau_f(\mathbf{u})|\mathbf{u}_T-\mathbf{u}_{TR}|\mathrm{d}\Gamma\geq 0. \tag{3.2}$$

Let us introduce the following notations for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}$,

$$a(\mathbf{w};\mathbf{u},\mathbf{v})=\int_\Omega \frac{2}{3}\frac{\sigma_{p1}(\mathbf{w})}{\dot{\bar\varepsilon}(\mathbf{w})}\dot\varepsilon_{ij}(\mathbf{u})\dot\varepsilon_{ij}(\mathbf{v})\mathrm{d}\mathbf{x}, \tag{3.3}$$

$$j_0(\mathbf{v})=\int_\Omega \sigma_{p0}\dot{\bar\varepsilon}(\mathbf{v})\mathrm{d}\mathbf{x},\quad j(\mathbf{u},\mathbf{v})=\int_{\Gamma_4}\tau_f(\mathbf{u})|\mathbf{v}_T-\mathbf{u}_{TR}|\mathrm{d}\Gamma, \tag{3.4}$$

then the variational statement of Problem 1 is as follows:

Problem 2 Find $\mathbf{u}\in\mathbf{W}$, satisfying

$$a(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u})+j_0(\mathbf{v})-j_0(\mathbf{u})+j(\mathbf{u},\mathbf{v})-j(\mathbf{u},\mathbf{u})\geq 0,\quad \forall \mathbf{v}\in\mathbf{W}. \tag{3.5}$$

Assuming further the following relations between the hydrostatic pressure and the volume dilatation strain-rate in Ω and between the normal stress and velocity on Γ_4

$$\sigma_H(\mathbf{u})=\frac{\dot\varepsilon_V(\mathbf{u})}{d},\quad \sigma_N(\mathbf{u})=-\frac{u_N}{d_N}, \tag{3.6}$$

where d and d_N are small positive penalty constants, and denoting

$$b(\mathbf{u};\mathbf{u},\mathbf{v})=a(\mathbf{u};\mathbf{u},\mathbf{v})+\int_\Omega \frac{1}{d}\dot\varepsilon_V(\mathbf{u})\dot\varepsilon_V(\mathbf{v})\mathrm{d}\mathbf{x}+\int_{\Gamma_4}\frac{1}{d_N}u_N v_N\mathrm{d}\Gamma, \tag{3.7}$$

we obtain the following variational penalty formulation of Problem 1:

Problem 3 Find $\mathbf{u}\in\mathbf{V}$, satisfying for all $\mathbf{v}\in\mathbf{V}$ the variational inequality

$$b(\mathbf{u};\mathbf{u},\mathbf{v}-\mathbf{u})+j_0(\mathbf{v})-j_0(\mathbf{u})+j(\mathbf{u},\mathbf{v})-j(\mathbf{u},\mathbf{u})\geq 0. \tag{3.8}$$

Since the functionals $j_0(\mathbf{v})$ and $j(\mathbf{u},\mathbf{v})$ are nondifferentiable at $\mathbf{v}=0$ and $\mathbf{v}_T=\mathbf{u}_{TR}$, we introduce the following convex, regularized functionals ([10, Chap. 10], [13]):

$$j_{0\epsilon}(\mathbf{v})=\int_\Omega \sigma_{p0}\sqrt{\dot{\bar\varepsilon}^2(\mathbf{v})+\epsilon^2}\mathrm{d}\mathbf{x},\quad j_{d_T}(\mathbf{u},\mathbf{v})=\int_{\Gamma_4}\tau_f(\mathbf{u})\sqrt{|\mathbf{v}_T-\mathbf{u}_{TR}|^2+d_T^2}\mathrm{d}\Gamma, \tag{3.9}$$

where $\epsilon>0$ and $d_T>0$ are constants. These functionals are already differentiable

$$\left\langle j_{0\epsilon}'(\mathbf{u}),\mathbf{v}\right\rangle=\int_\Omega \sigma_{p0}\frac{\frac{2}{3}\dot\varepsilon_{ij}(\mathbf{u})\dot\varepsilon_{ij}(\mathbf{v})}{\sqrt{\dot{\bar\varepsilon}^2(\mathbf{u})+\epsilon^2}}\mathrm{d}\mathbf{x}, \tag{3.10}$$

$$\left\langle j_{d_T}'(\mathbf{u},\mathbf{u}),\mathbf{v}\right\rangle=\int_{\Gamma_4}\tau_f(\mathbf{u})\frac{(\mathbf{u}_T-\mathbf{u}_{TR})\cdot\mathbf{v}_T}{\sqrt{|\mathbf{u}_T-\mathbf{u}_{TR}|^2+d_T^2}}\mathrm{d}\Gamma, \tag{3.11}$$

and such that

$$\left\langle j_{0\epsilon}^{'}(\mathbf{u}), \mathbf{v}-\mathbf{u}\right\rangle \le j_{0\epsilon}(\mathbf{v}) - j_{0\epsilon}(\mathbf{u}), \tag{3.12}$$

$$\left\langle j_{d_T}^{'}(\mathbf{u},\mathbf{u}), \mathbf{v}-\mathbf{u}\right\rangle \le j_{d_T}(\mathbf{u},\mathbf{v}) - j_{d_T}(\mathbf{u},\mathbf{u}), \tag{3.13}$$

$$\left\langle j_{0\epsilon}^{'}(\mathbf{v}) - j_{0\epsilon}^{'}(\mathbf{u}), \mathbf{v}-\mathbf{u}\right\rangle \ge 0, \tag{3.14}$$

$$\left\langle j_{d_T}^{'}(\mathbf{u},\mathbf{v}) - j_{d_T}^{'}(\mathbf{u},\mathbf{u}), \mathbf{v}-\mathbf{u}\right\rangle \ge 0. \tag{3.15}$$

Then we obtain the following regularized problem:

Problem $3^{\epsilon,d_T}$ Find $\mathbf{u}\in\mathbf{V}$, satisfying for all $\mathbf{v}\in\mathbf{V}$

$$b(\mathbf{u};\mathbf{u},\mathbf{v}) + \left\langle j_{0\epsilon}^{'}(\mathbf{u}),\mathbf{v}\right\rangle + \left\langle j_{d_T}^{'}(\mathbf{u},\mathbf{u}),\mathbf{v}\right\rangle = 0. \tag{3.16}$$

It is clear that the solutions of Problem 3 and Problem $3^{\epsilon,d_T}$ will depend on the introduced penalty and regularization constants. From (3.7), (3.3) and (2.4), it easily follows that there exist positive constants α_1 and α_2, such that

$$b(\mathbf{u};\mathbf{u},\mathbf{u}) \ge \alpha_1 \left(\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}\mathbf{x} + \int_{\Gamma_4} u_N u_N \mathrm{d}\Gamma\right), \tag{3.17a}$$

$$b(\mathbf{u};\mathbf{u},\mathbf{v}) \le \alpha_2 \left|\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{v})\mathrm{d}\mathbf{x} + \int_{\Gamma_4} u_N v_N \mathrm{d}\Gamma\right|. \tag{3.17b}$$

With the help of Korn's inequality [7, Chap. 3], [10, Chap. 5],

$$\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \int_\Omega u_i u_i \mathrm{d}x \ge c_K\|\mathbf{u}\|_1^2, \quad \forall \mathbf{u}\in (H^1(\Omega))^k, \tag{3.18}$$

where $c_K > 0$ is a constant, the following result holds:

Lemma 3.1 *There exists a constant $\beta > 0$, such that*

$$b(\mathbf{u};\mathbf{u},\mathbf{u}) \ge \beta\|\mathbf{u}\|_1^2, \quad \forall \mathbf{u}\in\mathbf{V}. \tag{3.19}$$

Proof From (3.17a) and (3.18) it follows that (3.19) will be satisfied if we prove that

$$\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \int_{\Gamma_4} u_N u_N \mathrm{d}\Gamma \ge \beta_1 \int_\Omega u_i u_i \mathrm{d}x,$$

where $\beta_1 > 0$ is a constant. The case $\mathbf{u} = 0$ is trivial. For $\mathbf{u} \ne 0$, without loss of generality we substitute $\mathbf{u}$ by $\mathbf{u}/\|\mathbf{u}\|_0$ and using the same notation, since $\|\mathbf{u}\|_0 = 1$, we have to prove that

$$\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \int_{\Gamma_4} u_N u_N \mathrm{d}\Gamma \ge \beta_1.$$

We suppose the contrary. Then there exists a sequence $\{\mathbf{u}_n\}\in\mathbf{V}$, such that $\|\mathbf{u}_n\|_0 = 1$ and

$$\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u}_n)\dot{\varepsilon}_{ij}(\mathbf{u}_n)\mathrm{d}x + \int_{\Gamma_4} u_{Nn} u_{Nn} \mathrm{d}\Gamma \underset{n\to\infty}{\longrightarrow} 0.$$

From (3.18) it then follows that $\|\mathbf{u}_n\|_1 <$ const.. Therefore there exists a subsequence of $\{\mathbf{u}_n\}$, also denoted by $\{\mathbf{u}_n\}$, which is weakly convergent to $\mathbf{u}\in(H^1(\Omega))^k$. Then we have that

$$0 = \liminf_{n\to\infty}\left(\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u}_n)\dot{\varepsilon}_{ij}(\mathbf{u}_n)\mathrm{d}x + \int_{\Gamma_4} u_{Nn}u_{Nn}\mathrm{d}\Gamma\right) \ge \int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \int_{\Gamma_4} u_N u_N \mathrm{d}\Gamma \ge 0,$$

and hence

$$\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x = 0, \quad \int_{\Gamma_4} u_N u_N \mathrm{d}\Gamma = 0,$$

i.e., $\dot{\varepsilon}_{ij}(\mathbf{u}) = 0$ in Ω and $u_N = 0$ on Γ_4. But we also have $u_N = 0$ on Γ_3 and Γ_3 is not parallel to Γ_4, which yields $\mathbf{u} = \mathbf{0}$ in $\Omega \cup \Gamma$. Further, since $\mathbf{V}$ is compactly embedded in $\mathbf{H}$, from the weak convergence of $\{\mathbf{u}_n\}$ in $\mathbf{V}$ follows its strong convergence in $\mathbf{H}$, i.e., $\{\mathbf{u}_n\}\underset{n\to\infty}{\longrightarrow}0$ in $\mathbf{H}$, which contradicts our assumption $\|\mathbf{u}\|_0 = 1$. Then we have

$$
\begin{aligned}
b(\mathbf{u}; \mathbf{u}, \mathbf{u}) &\geq \alpha_1 \left(\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \int_{\Gamma_4} u_N u_N \mathrm{d}\Gamma \right) \\
&\geq \frac{\alpha_1}{2} \int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \frac{\alpha_1}{2} \left(\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \int_{\Gamma_4} u_N u_N \mathrm{d}\Gamma \right) \\
&\geq \frac{\alpha_1}{2} \left(\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \beta_1 \int_\Omega u_i u_i \mathrm{d}x \right) \\
&\geq \min\left(\frac{\alpha_1}{2}, \frac{\alpha_1\beta_1}{2} \right) \left(\int_\Omega \dot{\varepsilon}_{ij}(\mathbf{u})\dot{\varepsilon}_{ij}(\mathbf{u})\mathrm{d}x + \int_\Omega u_i u_i \mathrm{d}x \right) \geq \beta \|\mathbf{u}\|_1^2,
\end{aligned}
$$

which completes the proof. □

Remark 3.1 The above lemma also states that the norm $\|\mathbf{u}\|_V$ is equivalent to the $(H^1(\Omega))^k$ norm $\|\mathbf{u}\|_1$. It can be further shown that, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$, the following properties of the functionals above hold ([5]):

$$b(\mathbf{u}; \mathbf{u}, \mathbf{u}) \geq c_1 \|\mathbf{u}\|_V^2, \quad b(\mathbf{u}; \mathbf{v}, \mathbf{w}) \leq c_2 \|\mathbf{v}\|_V \|\mathbf{w}\|_V, \tag{3.20}$$

$$b(\mathbf{v}; \mathbf{v}, \mathbf{v} - \mathbf{u}) - b(\mathbf{u}; \mathbf{u}, \mathbf{v} - \mathbf{u}) \geq m \|\mathbf{v} - \mathbf{u}\|_V^2, \tag{3.21}$$

$$b(\mathbf{v}; \mathbf{v}, \mathbf{w}) - b(\mathbf{u}; \mathbf{u}, \mathbf{w}) \leq M \|\mathbf{v} - \mathbf{u}\|_V \|\mathbf{w}\|_V, \tag{3.22}$$

$$0 \leq j_0(\mathbf{u}) \leq c_3 \|\mathbf{u}\|_V, \quad 0 \leq j(\mathbf{u}, \mathbf{v}) \leq c_4 \|\mathbf{u}\|_V \|\mathbf{v}_T - \mathbf{u}_{TR}\|_V, \tag{3.23}$$

$$|j_0(\mathbf{v}) - j_0(\mathbf{u})| \leq c_0 \|\mathbf{v} - \mathbf{u}\|_V, \tag{3.24}$$

$$|j(\mathbf{u}, \mathbf{w}) + j(\mathbf{w}, \mathbf{v}) - j(\mathbf{u}, \mathbf{v}) - j(\mathbf{w}, \mathbf{w})| \leq c \|\mathbf{w} - \mathbf{u}\|_V \|\mathbf{w} - \mathbf{v}\|_V, \tag{3.25}$$

where $c_0, c_1, c_2, c_3, c_4, c, m$ and M are positive constants, as c and c_3 depend on the friction coefficient $\mu_f(\mathbf{x}) \in L_\infty(\Gamma_4)$. □

4 Existence, uniquenes and convergence

We shall now briefly present an algorithmically oriented proof of the following existence and uniqueness theorem, proved in [5] in a general setting:

Theorem 4.1 *Let the properties given in Remark 3.1 hold. Then, for a sufficiently small friction coefficient, Problem 3 has a unique solution* $\mathbf{u} \in \mathbf{V}$.

Sketch of proof The existence part is based on proving the convergence of the successive iteration (secant-modulus) method, defined by the following.

Problem 3$_n$ For an arbitrary initial $\mathbf{u}_0 \in \mathbf{V}$ find $\mathbf{u}_{n+1}$, $n = 0, 1, \ldots$ satisfying for all $\mathbf{v} \in \mathbf{V}$

$$b(\mathbf{u}_n; \mathbf{u}_{n+1}, \mathbf{v} - \mathbf{u}_{n+1}) + j_0(\mathbf{v}) - j_0(\mathbf{u}_{n+1}) + j(\mathbf{u}_n, \mathbf{v}) - j(\mathbf{u}_n, \mathbf{u}_{n+1}) \geq 0. \tag{4.1}$$

Since this problem has unique solutions $\mathbf{u}_{n+1}$ for every $n = 0, 1, 2, \ldots$, ([5], [14, App. 1], [15, Chap. 1]), and for sufficiently small coefficient of friction, the sequence $\{\mathbf{u}_n\}$ is such that

$$\|\mathbf{u}_{n+1} - \mathbf{u}_n\|_V \leq \cdots \leq q^n \|\mathbf{u}_1 - \mathbf{u}_0\|_V, \quad 0 < q < 1, \quad \text{i.e.,} \lim_{n\to\infty} \|\mathbf{u}_{n+1} - \mathbf{u}_n\|_V = 0, \tag{4.2}$$

it follows that it is fundamental and therefore there exists an unique $\mathbf{u} \in \mathbf{V}$ to which $\mathbf{u}_n$ converges strongly as $n \to \infty$ in $\mathbf{V}$. This $\mathbf{u}$ is a solution of Problem 3, since after $\mathbf{v}$ is replaced with $\mathbf{u}_n$ in Problem 3 and $\mathbf{v}$ in Problem 3_n is replaced, respectively, with $\mathbf{u}$ and $\mathbf{u}_n$, it consequently follows that

$$\begin{aligned}
&b(\mathbf{u};\mathbf{u},\mathbf{u}-\mathbf{u}_n) - b(\mathbf{u}_n;\mathbf{u}_n,\mathbf{u}-\mathbf{u}_n)\\
&\quad\le j_0(\mathbf{u}_n) - j_0(\mathbf{u}) + j(\mathbf{u},\mathbf{u}_n) - j(\mathbf{u},\mathbf{u}) - b(\mathbf{u}_n;\mathbf{u}_n,\mathbf{u}-\mathbf{u}_n)\\
&\qquad + b(\mathbf{u}_n;\mathbf{u}_{n+1},\mathbf{u}-\mathbf{u}_n) - b(\mathbf{u}_n;\mathbf{u}_{n+1},\mathbf{u}-\mathbf{u}_n)\\
&\quad= b(\mathbf{u}_n;\mathbf{u}_{n+1}-\mathbf{u}_n,\mathbf{u}-\mathbf{u}_n) + j_0(\mathbf{u}_n) - j_0(\mathbf{u}) + j(\mathbf{u},\mathbf{u}_n) - j(\mathbf{u},\mathbf{u})\\
&\qquad + b(\mathbf{u}_n;\mathbf{u}_{n+1},\mathbf{u}_n-\mathbf{u}_{n+1}) - b(\mathbf{u}_n;\mathbf{u}_{n+1},\mathbf{u}-\mathbf{u}_{n+1})\\
&\quad\le b(\mathbf{u}_n;\mathbf{u}_{n+1}-\mathbf{u}_n,\mathbf{u}-\mathbf{u}_n) + j(\mathbf{u},\mathbf{u}_n) - j(\mathbf{u},\mathbf{u}) + j(\mathbf{u}_n,\mathbf{u}) - j(\mathbf{u}_n,\mathbf{u}_n)\\
&\qquad + b(\mathbf{u}_n;\mathbf{u}_{n+1},\mathbf{u}_n-\mathbf{u}_{n+1}) + j_0(\mathbf{u}_n) - j_0(\mathbf{u}_{n+1}) + j(\mathbf{u}_n,\mathbf{u}_n) - j(\mathbf{u}_n,\mathbf{u}_{n+1}).
\end{aligned} \tag{4.3}$$

Using the inequalities in Remark 3.1, it further follows that

$$\begin{aligned}
&(m-c)\|\mathbf{u}-\mathbf{u}_n\|_V^2 \le C_1\|\mathbf{u}-\mathbf{u}_n\|_V\|\mathbf{u}_n-\mathbf{u}_{n+1}\|_V\\
&\quad + C_2\|\mathbf{u}_{n+1}\|_V\|\mathbf{u}_{n+1}-\mathbf{u}_n\|_V + C_3\|\mathbf{u}_{n+1}-\mathbf{u}_n\|_V + C_4\|\mathbf{u}_n\|_V\|\mathbf{u}_{n+1}-\mathbf{u}_n\|_V,
\end{aligned} \tag{4.4}$$

where C_1, C_2, C_3, C_4 are positive constants. Since for sufficiently small coefficient of friction $c < m$ and since $\{\mathbf{u}_n\}$ is bounded in $\mathbf{V}$, for $n \to \infty$ it follows that

$$\lim_{n\to\infty} \|\mathbf{u}-\mathbf{u}_n\|_V = 0. \quad \square \tag{4.5}$$

Next we shall study the convergence properties of the sequence of solutions $\{\mathbf{u}^{d,d_N}\}$ of Problem 3, obtained for all sufficiently small penalty constants $d > 0$ and $d_N > 0$ ([5], [10, Chap. 7], [16, Chap. 3, Sect. 5]). This sequence is bounded in $\mathbf{V}$, since setting $\mathbf{v} = \mathbf{0}$ in (3.8), we have $\|\mathbf{u}\|_V \le c_f|\mathbf{u}_{TR}|$, where c_f is a positive constant, depending on the coefficient of friction. Taking then, without loss of generality, $d_N = c_N d$, where c_N is a positive constant, we can construct by diagonalization a subsequence $\{\mathbf{u}^d\}$, which is weakly convergent in $\mathbf{V}$, such that the following result holds:

Theorem 4.2 *At $d \to 0$ there exists an element $\mathbf{u} \in \mathbf{W}$, which is the unique solution of the Problem 2.*

Proof For $\mathbf{u}^d \in \mathbf{V}$ and all $\mathbf{v} \in \mathbf{V}$, from (3.8) it follows

$$\begin{aligned}
&\Big[b(\mathbf{u}^d;\mathbf{u}^d,\mathbf{u}^d) + j_0(\mathbf{u}^d) + j(\mathbf{u}^d,\mathbf{u}^d)\Big]\\
&\quad -\frac{1}{d}\int_\Omega \dot{\varepsilon}_V(\mathbf{u}^d)\dot{\varepsilon}_V(\mathbf{v})\mathrm{d}x - \frac{1}{c_N d}\int_{\Gamma_4} u_N^d v_N \mathrm{d}\Gamma \le a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v}) + j(\mathbf{u}^d,\mathbf{v}) + j_0(\mathbf{v}).
\end{aligned} \tag{4.6}$$

Since the quantity in brackets in the left-hand side of (4.6) is nonnegative, we have

$$-c_N\int_\Omega \dot{\varepsilon}_V(\mathbf{u}^d)\dot{\varepsilon}_V(\mathbf{v})\mathrm{d}x - \int_{\Gamma_4} u_N^d v_N \mathrm{d}\Gamma \le c_N d\Big[|a(\mathbf{u}^d;\mathbf{u}^d,\mathbf{v})| + j(\mathbf{u}^d,\mathbf{v}) + j_0(\mathbf{v})\Big]. \tag{4.7}$$

Having in mind that the right-hand side of (4.6) is bounded and since from the weak convergence of $\mathbf{u}^d$ in $\mathbf{V}$ it follows that

$$\dot{\varepsilon}_V(\mathbf{u}^d) \to \dot{\varepsilon}_V(\mathbf{u}) \quad \text{weakly in } \mathbf{H}, \tag{4.8}$$

$$u_N^d \to u_N \quad \text{weakly in } H^{1/2}(\Gamma_4), \tag{4.9}$$

taking $d \to 0$ in (4.7), we obtain for all $+\mathbf{v}, -\mathbf{v} \in \mathbf{V}$,

$$-c_N\int_\Omega \dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v})\mathrm{d}x - \int_{\Gamma_4} u_N v_N \mathrm{d}\Gamma \le 0, \tag{4.10}$$

$$+c_N\int_\Omega \dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v})\mathrm{d}x + \int_{\Gamma_4} u_N v_N \mathrm{d}\Gamma \le 0. \tag{4.11}$$

Therefore we have

$$\int_\Omega \dot{\varepsilon}_V(\mathbf{u})\dot{\varepsilon}_V(\mathbf{v})\mathrm{d}x \equiv 0, \quad \int_{\Gamma_4} u_N v_N \mathrm{d}\Gamma \equiv 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{4.12}$$

and hence $\dot{\varepsilon}_V(\mathbf{u}) \equiv 0$ and $u_N \equiv 0$, i.e., $\mathbf{u} \in \mathbf{W}$. We shall now show that this $\mathbf{u}$ is a solution of (3.5) in $\mathbf{W}$. Since for all $\mathbf{w} \in \mathbf{W}$ we have

$$\begin{aligned}
&a(\mathbf{w}; \mathbf{w}, \mathbf{w} - \mathbf{u}^d) + j_0(\mathbf{w}) - j_0(\mathbf{u}^d) + j(\mathbf{w}, \mathbf{w}) - j(\mathbf{w}, \mathbf{u}^d)\\
&\quad = a(\mathbf{w}; \mathbf{w}, \mathbf{w} - \mathbf{u}^d) + j_0(\mathbf{w}) - j_0(\mathbf{u}^d) + j(\mathbf{w}, \mathbf{w}) - j(\mathbf{w}, \mathbf{u}^d)\\
&\qquad - \Big[b(\mathbf{u}^d; \mathbf{u}^d, \mathbf{w} - \mathbf{u}^d) + j_0(\mathbf{w}) - j_0(\mathbf{u}^d) + j(\mathbf{u}^d, \mathbf{w}) - j(\mathbf{u}^d, \mathbf{u}^d)\Big]\\
&\qquad + \Big[b(\mathbf{u}^d; \mathbf{u}^d, \mathbf{w} - \mathbf{u}^d) + j_0(\mathbf{w}) - j_0(\mathbf{u}^d) + j(\mathbf{u}^d, \mathbf{w}) - j(\mathbf{u}^d, \mathbf{u}^d)\Big]\\
&\quad \geq (m - c)\|\mathbf{w} - \mathbf{u}^d\|_V^2 \geq 0,
\end{aligned} \tag{4.13}$$

taking $d \to 0$ we obtain

$$a(\mathbf{w}; \mathbf{w}, \mathbf{w} - \mathbf{u}) + j_0(\mathbf{w}) - j_0(\mathbf{u}) + j(\mathbf{w}, \mathbf{w}) - j(\mathbf{w}, \mathbf{u}) \geq 0, \quad \forall \mathbf{w} \in \mathbf{W}. \tag{4.14}$$

Setting $\mathbf{w} = \mathbf{u} + t(\mathbf{v} - \mathbf{u}))$, $t \in [0, 1]$, $\forall \mathbf{v} \in \mathbf{W}$, we obtain

$$\begin{aligned}
0 &\leq a(\mathbf{u} + t(\mathbf{v} - \mathbf{u}); \mathbf{u} + t(\mathbf{v} - \mathbf{u}), t(\mathbf{v} - \mathbf{u})) + j_0(\mathbf{u} + t(\mathbf{v} - \mathbf{u})) - j_0(\mathbf{u})\\
&\quad + j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{u} + t(\mathbf{v} - \mathbf{u})) - j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{u})\\
&\leq t a(\mathbf{u} + t(\mathbf{v} - \mathbf{u}); \mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{v} - \mathbf{u}) + (1 - t) j_0(\mathbf{u}) + t j_0(\mathbf{v}) - j_0(\mathbf{u})\\
&\quad + (1 - t) j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{u}) + t j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{v}) - j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{u})\\
&= t a(\mathbf{u} + t(\mathbf{v} - \mathbf{u}); \mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{v} - \mathbf{u}) + t j_0(\mathbf{v}) - t j_0(\mathbf{u})\\
&\quad + t j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{v}) - t j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{u}).
\end{aligned} \tag{4.15}$$

Hence for $t \neq 0$ we have that

$$\begin{aligned}
&a(\mathbf{u} + t(\mathbf{v} - \mathbf{u}); \mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{v} - \mathbf{u}) + j_0(\mathbf{v}) - j_0(\mathbf{u})\\
&\quad + j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{v}) - j(\mathbf{u} + t(\mathbf{v} - \mathbf{u}), \mathbf{u}) \geq 0
\end{aligned} \tag{4.16}$$

and taking $t \to 0$ we finally obtain

$$a(\mathbf{u}; \mathbf{u}, \mathbf{v} - \mathbf{u}) + j_0(\mathbf{v}) - j_0(\mathbf{u}) + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \geq 0, \quad \forall \mathbf{v} \in \mathbf{W}, \tag{4.17}$$

which is Problem 2. In order to establish the uniqueness of $\mathbf{u}$, let us assume that $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{W}$ are two different solutions, i.e.,

$$a(\mathbf{u}_1; \mathbf{u}_1, \mathbf{v} - \mathbf{u}_1) + j_0(\mathbf{v}) - j_0(\mathbf{u}_1) + j(\mathbf{u}_1, \mathbf{v}) - j(\mathbf{u}_1, \mathbf{u}_1) \geq 0 \tag{4.18}$$

$$a(\mathbf{u}_2; \mathbf{u}_2, \mathbf{v} - \mathbf{u}_2) + j_0(\mathbf{v}) - j_0(\mathbf{u}_2) + j(\mathbf{u}_2, \mathbf{v}) - j(\mathbf{u}_2, \mathbf{u}_2) \geq 0. \tag{4.19}$$

Setting $\mathbf{v} = \mathbf{u}_2$ in (4.18) and $\mathbf{v} = \mathbf{u}_1$ in (4.18), after adding the inequalities and rearranging we obtain

$$j(\mathbf{u}_1, \mathbf{u}_2) + j(\mathbf{u}_2, \mathbf{u}_1) - j(\mathbf{u}_1, \mathbf{u}_1) - j(\mathbf{u}_2, \mathbf{u}_2) \geq a(\mathbf{u}_1; \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - a(\mathbf{u}_2; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2). \tag{4.20}$$

Using Remark 3.1, we obtain that for a sufficiently small coefficient of friction, i.e., for $c < m$,

$$0 \geq (m - c)\|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 > 0, \tag{4.21}$$

which yields $\mathbf{u}_1 \equiv \mathbf{u}_2$. □

Remark 4.1 Existence and uniqueness of the solution of Problem $3^{\epsilon,d_T}$ can be proved analogously to Problem 3. □

Further we shall study the convergence properties of the sequence of solutions $\{\mathbf{u}^{\epsilon,d_T}\}$ of Problem $3^{\epsilon,d_T}$, obtained for all sufficiently small regularization constants $d_T > 0$ and $\epsilon > 0$ ([10, Chap. 10], [13]). This sequence is bounded in $\mathbf{V}$ and taking without loss of generality, $d_T = c_T\epsilon$, where c_T is a positive constant, we can construct by diagonalization a subsequence $\{\mathbf{u}^\epsilon\}$, weakly convergent in $\mathbf{V}$, such that the following result is obtained:

Theorem 4.3 *Let* $\mathbf{u} \in \mathbf{V}$ *and* $\mathbf{u}^\epsilon \in \mathbf{V}$ *be the solutions of the Problem 3 and Problem* $3^{\epsilon,d_T}$*, respectively. Then there exists a positive constant* C_0*, independent of* ϵ*, such that*

$$\|\mathbf{u} - \mathbf{u}^\epsilon\|_V \le C_0\sqrt{\epsilon}. \tag{4.22}$$

Proof Let us set $\mathbf{v} = \mathbf{u}^\epsilon$ in (3.8) and $\mathbf{v} = \mathbf{u} - \mathbf{u}^\epsilon$ in (3.16). Then adding (3.8) to (3.16) and taking into account (3.12–3.15) and Remark 3.1, we have

$$\begin{aligned} m\|\mathbf{u} - \mathbf{u}^\epsilon\|_V^2 &\le b(\mathbf{u}; \mathbf{u}, \mathbf{u} - \mathbf{u}^\epsilon) - b(\mathbf{u}^\epsilon; \mathbf{u}^\epsilon, \mathbf{u} - \mathbf{u}^\epsilon) \\ &\le j_0(\mathbf{u}^\epsilon) - j_0(\mathbf{u}) + \langle j_{0\epsilon}'(\mathbf{u}^\epsilon, \mathbf{u} - \mathbf{u}^\epsilon\rangle \\ &\quad + j(\mathbf{u}, \mathbf{u}^\epsilon) - j(\mathbf{u}, \mathbf{u}) + \langle j_{d_T}'(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon), \mathbf{u} - \mathbf{u}^\epsilon\rangle \\ &\le j_0(\mathbf{u}^\epsilon) - j_0(\mathbf{u}) + j_{0\epsilon}(\mathbf{u}) - j_{0\epsilon}(\mathbf{u}^\epsilon) \\ &\quad + j(\mathbf{u}, \mathbf{u}^\epsilon) - j(\mathbf{u}, \mathbf{u}) + j_{d_T}(\mathbf{u}^\epsilon, \mathbf{u}) - j_{d_T}(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon) \\ &\quad + j(\mathbf{u}^\epsilon, \mathbf{u}) - j(\mathbf{u}^\epsilon, \mathbf{u}) + j(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon) - j(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon) \le C_1\epsilon + C_2\|\mathbf{u} - \mathbf{u}^\epsilon\|_V^2, \end{aligned} \tag{4.23}$$

where C_1 and C_2 are positive constants. For a sufficiently small friction coefficient, such that $m > C_2$, there exists a positive constant C_0 independent of ϵ for which

$$\|\mathbf{u} - \mathbf{u}^\epsilon\|_V \le \sqrt{\frac{C_1\epsilon}{m - C_2}} = C_0\sqrt{\epsilon}. \quad \square \tag{4.24}$$

5 Finite-element approximation

Let $\mathcal{C}_h$ be a regular partitioning of $\bar{\Omega} = \cup_{K\in\mathcal{C}_h} K$ into finite elements K and let us construct the finite-element spaces

$$\mathbf{V}_h = \left\{\mathbf{v}^h : \mathbf{v}^h \in \mathbf{V} \cap (C^0(\bar{\Omega}))^k, \quad \mathbf{v}^h{}_{|K} = \hat{\mathbf{v}}^h \circ F_K^{-1}, \quad \hat{\mathbf{v}}_h \in (Q_1(\hat{K}))^k\right\},$$

where h is the mesh parameter approaching zero, $F_K : \hat{K} \to K$, $F_K \in (Q_1(\hat{K}))^k$ is the bilinear isoparametric transformation, $\hat{K}$ is the reference element and $(Q_1(\hat{K}))^k$ is the space of polynomials on $\hat{K}$ of order not greater than one in each variable. Let us also suppose that the following standard approximation properties of $\mathbf{V}_h$ hold [10, Chap. 4], [17, Chap. 3]:

$$\begin{aligned} &\forall \mathbf{v} \in (H^m(\Omega))^k \cap \mathbf{V}, \quad \exists \mathbf{v}^h \in \mathbf{V}_h, \text{ such that} \\ &\|\mathbf{v} - \mathbf{v}^h\|_s \le c_\Omega h^r \|\mathbf{v}\|_m, \quad r = \min\{2 - s, m - s\}, \quad m > s \ge 0, \\ &\text{and if } \boldsymbol{\gamma}_0(\mathbf{v}) \in (H^p(\Gamma))^k, \text{ then} \\ &\|\boldsymbol{\gamma}_0(\mathbf{v}) - \boldsymbol{\gamma}_0(\mathbf{v}^h)\|_{q,\Gamma} \le c_\Gamma h^{r_1} \|\boldsymbol{\gamma}_0(\mathbf{v})\|_{p,\Gamma}, \quad r_1 = \min\{3/2 - q, p - q\}, \quad m > p > q, \end{aligned} \tag{5.1}$$

where $c_\Omega > 0$ and c_Γ are constants that are independent of h and $\mathbf{v}$. Then from Problem $3^{\epsilon,d_T}$ we obtain in $\mathbf{V}_h$ the following finite-dimensional problem:

Problem $\mathbf{3}_h^{\epsilon,d_T}$ Find $\mathbf{u}^h \in \mathbf{V}_h$, satisfying for all $\mathbf{v}^h \in \mathbf{V}_h$ the equation

$$b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h) + \langle j_{0\epsilon}'(\mathbf{u}^h), \mathbf{v}^h\rangle + \langle j_{d_T}'(\mathbf{u}^h, \mathbf{u}^h), \mathbf{v}^h\rangle = 0. \tag{5.2}$$

Theorem 5.1 *Let* $\mathbf{u} \in (H^2(\Omega))^k \cap \mathbf{V}$ *and* $\mathbf{u}^h \in \mathbf{V}_h$ *be the solutions of the Problem* $3^{\epsilon,d_T}$ *and Problem* $3_h^{\epsilon,d_T}$*, respectively. Then there exists a positive constant C, independent of h, such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_V \leq C\|\mathbf{u}\|_2 h. \tag{5.3}$$

Proof Subtracting (5.2) from (3.16) with $\mathbf{v} = \mathbf{v}^h$ we obtain

$$b(\mathbf{u}; \mathbf{u}, \mathbf{v}^h) - b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h) + \langle j'_{0\epsilon}(\mathbf{u}), \mathbf{v}^h\rangle - \langle j'_{0\epsilon}(\mathbf{u}^h), \mathbf{v}^h\rangle + \langle j'_{d_T}(\mathbf{u}, \mathbf{u}), \mathbf{v}^h\rangle - \langle j'_{d_T}(\mathbf{u}^h, \mathbf{u}^h), \mathbf{v}^h\rangle = 0. \tag{5.4}$$

Replacing further $\mathbf{v}^h$ with $(\mathbf{u}-\mathbf{u}^h) - (\mathbf{u}-\mathbf{v}^h)$ and taking into account Remark 3.1 and (3.12–3.15), we consequently have

$$\begin{aligned} m\|\mathbf{u} - \mathbf{u}^h\|_V^2 &\leq b(\mathbf{u}; \mathbf{u}, \mathbf{u}-\mathbf{u}^h) - b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{u}-\mathbf{u}^h) \\ &\quad + \langle j'_{0\epsilon}(\mathbf{u}) - j'_{0\epsilon}(\mathbf{u}^h), \mathbf{u}-\mathbf{u}^h\rangle + \langle j'_{d_T}(\mathbf{u}, \mathbf{u}) - j'_{d_T}(\mathbf{u}, \mathbf{u}^h), \mathbf{u}-\mathbf{u}^h\rangle \\ &= b(\mathbf{u}; \mathbf{u}, \mathbf{u}-\mathbf{v}^h) - b(\mathbf{u}^h; \mathbf{u}^h, \mathbf{u}-\mathbf{v}^h) + \langle j'_{0\epsilon}(\mathbf{u}) - j'_{0\epsilon}(\mathbf{u}^h), \mathbf{u}-\mathbf{v}^h\rangle \\ &\quad + \langle j'_{d_T}(\mathbf{u}^h, \mathbf{u}^h) - j'_{d_T}(\mathbf{u}, \mathbf{u}^h), \mathbf{u}-\mathbf{u}^h\rangle + \langle j'_{d_T}(\mathbf{u}, \mathbf{u}) - j'_{d_T}(\mathbf{u}^h, \mathbf{u}^h), \mathbf{u}-\mathbf{v}^h\rangle \\ &\leq C_1\|\mathbf{u}-\mathbf{u}^h\|_V\|\mathbf{u}-\mathbf{v}^h\|_V + C_2\|\mathbf{u}-\mathbf{u}^h\|_V^2, \end{aligned} \tag{5.5}$$

where C_1 and C_2 are positive constants. Then, for a sufficiently small friction coefficient, such that $m > C_2$, we obtain

$$\|\mathbf{u} - \mathbf{u}^h\|_V \leq \frac{C_1}{m - C_2}\|\mathbf{u} - \mathbf{v}^h\|_V; \tag{5.6}$$

after taking into account the finite-element interpolation properties (5.1), we conclude that there exists a positive constant C independent of h, such that (5.3) holds. □

Remark 5.1 Existence and uniqueness results, for the discrete problem considered here, can be obtained analogously to the continuous problem of the preceding section. □

6 Algorithm and numerical results

Applying the secant-modulus method to the Problem $3_h^{\epsilon,d_T}$, we obtain:

Problem $\mathbf{3}_{h,n}^{\epsilon,d_T}$ Find $\mathbf{u}_{n+1}^h \in \mathbf{V}_h$, $n = 0, 1, 2, \ldots$, satisfying for arbitrary initial $\mathbf{u}_0^h \in \mathbf{V}_h$ and every $\mathbf{v}^h \in \mathbf{V}_h$ the equation

$$b(\mathbf{u}_n^h; \mathbf{u}_{n+1}^h, \mathbf{v}^h) + \langle j'_{0\epsilon}(\mathbf{u}_{n+1}^h), \mathbf{v}^h\rangle + \langle j'_{d_T}(\mathbf{u}_n^h, \mathbf{u}_{n+1}^h), \mathbf{v}^h\rangle = 0, \tag{6.1}$$

until $\|\mathbf{u}_{n+1}^h - \mathbf{u}_n^h\|/\|\mathbf{u}_{n+1}^h\| < \delta$, where $\|\cdot\|$ is a vector norm and δ is the accuracy tolerance.

This problem, using complete Gauss integration on every finite element, defines:

Algorithm 1 Find $\{u_{n+1}^h\}$, $n = 0, 1, 2, \ldots$, satisfying for arbitrary initial $\{u_0^h\}$ the system of equations

$$\mathbf{K}(u_n^h)\{u_{n+1}^h\} = \mathbf{F}(u_n^h), \tag{6.2}$$

until $\|u_{n+1}^h - u_n^h\|/\|u_{n+1}^h\| < \delta$.

Here $\mathbf{K}$ and $\mathbf{F}$ are the velocity-dependent stiffness matrix and the load vector. The vector of nodal velocities is denoted by $\{u_{n+1}^h\}$. We apply this algorithms to solve the following example problem [3]:

Example A two-dimensional workpiece with length 15 mm, initial and exit thicknesses 2 mm and 1.2 mm, respectively, is rolled with rolls with diameter 400 mm and velocity $\mathbf{u}_{TR} = 1256.64$ mm/s. The following empirical yield limit expression (2.3), satisfying (2.4) for all $\dot{\bar{\varepsilon}} \in [0, \infty)$, is supposed to hold:

$$\sigma_{p0} = \text{const.}, \quad \text{and } \sigma_{p1}(\dot{\bar{\varepsilon}}) = A\dot{\bar{\varepsilon}}^{\alpha} \quad \text{for } \dot{\bar{\varepsilon}} \in [\dot{\bar{\varepsilon}}_1, \dot{\bar{\varepsilon}}_2], \tag{6.3a}$$

$$\sigma_{p1}(\dot{\bar{\varepsilon}}) = \frac{\sigma_{p1}(\dot{\bar{\varepsilon}}_q)}{\dot{\bar{\varepsilon}}_q}\dot{\bar{\varepsilon}}, \quad q = 1, 2, \quad \text{for } \dot{\bar{\varepsilon}} \in [0, \dot{\bar{\varepsilon}}_1] \cup [\dot{\bar{\varepsilon}}_2, \infty), \tag{6.3b}$$

where $A > 0$, $\alpha \in (0, 1]$, $\dot{\bar{\varepsilon}}_1$ and $\dot{\bar{\varepsilon}}_2$ are material constants, depending on the process conditions. The following values are chosen for the material constants: $\sigma_{p0} = 1.0$ MPa, $A = 10^{-6}$, $\alpha = 1.0$, or $\sigma_{p0} = 0.0$, $A = 1.0$ MPa, $\alpha = 0.001$; $\dot{\bar{\varepsilon}}_1 = 10^{-3}$ and $\dot{\bar{\varepsilon}}_2 = 10^3$. Also the following values are taken for the friction coefficient $\mu = 0.1$ and for the regularization and penalty constants, respectively, $\epsilon = d_T = 10^{-6}$ and $d = d_N = 10^{-2} \approx 10^{-3}/\max(\sigma_p^h/\dot{\bar{\varepsilon}}^h)$.

Two finite-element meshes are constructed: an initial 30×3 coarse mesh and a final 60×4 fine mesh, containing correspondingly 90 and 240 quadrilateral finite elements. The contact boundary is discretized correspondingly by 20 and 40 bilinear line elements. The total number of nodes is 124 and 305 correspondingly. The computed values for the effective strain rates, normal and hydrostatic pressure and friction stresses are averaged at the finite-element nodes (centers). The computational experiments show that the algorithm is fast: results are obtained for 11 and 18 iterations, depending on the used mesh, within an accuracy of $\delta = 10^{-3}$. The values of the regularization and penalty constants are chosen optimally, in the sense that a further decrease leads to computational fluctuations and overconstraining. This shows the mesh-dependence of the penalty parameters, which also means that the external unknowns, normal and hydrostatic pressure, satisfy mesh-dependent stability condition. Therefore, despite the obtained theoretical convergence results for the continuous penalty method, if the penalty constants are not carefully chosen in relation to the mesh used, the discrete penalty method may not work. Further, it should be mentioned that, for both choices of the yield limit, the computed results are almost identical. The obtained normal pressures and friction stresses along Γ_4 and hydrostatic pressure and effective strain-rate distributions in Ω are illustrated in Figs. 2–4. The normal pressure and friction-stress results are very close to the corresponding results presented in [3], which finally supports the applicability and the effectiveness of the considered method of approach for solving rolling problems.

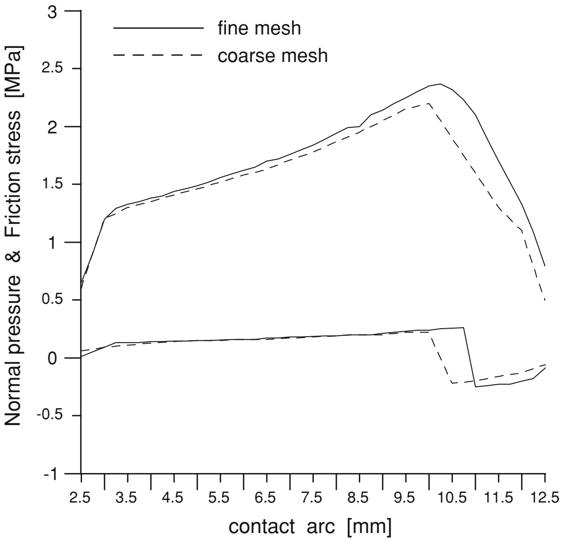

Fig. 2 Normal pressures $\bar{\sigma}_N$ (MPa) and frictional stresses $\boldsymbol{\sigma}_T$ (MPa) along Γ_4

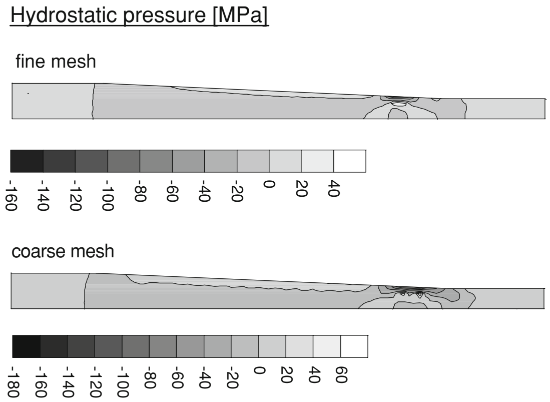

Fig. 3 Distribution of hydrostatic pressure σ_H (MPa) in Ω

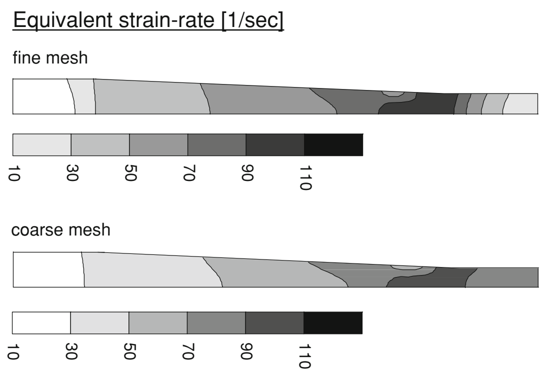

Fig. 4 Distribution of equivalent strain-rate $\dot{\bar{\varepsilon}}\,(\text{sec}^{-1})$ in Ω

7 Concluding remarks

We have considered an important class of contact problems in plasticity flow theory with nonlocal Coulomb friction, describing an isothermal, steady-state rolling process of a rigid–plastic, incompressible, strain-rate dependent metallic body by absolutely rigid rolls. The corresponding primal, penalty and regularized penalty variational formulations, in the form of strongly nonlinear variational inequalities and equations, are derived and analysed variationally and numerically. The fundamental work on elastic frictional-contact problems, variational inequalities and their numerical solution, by Duvaut and Lions [7, Chap. 3], Nečas and Hlavaček [8, Chap. 13], Panagiotopoulos [9, Chap. 5], Kikuchi and Oden [10, Chaps. 2–7, 10, 11], Lions [16, Chaps. 2, 3], Glowinski et al. [14, App. 1], Glowinski [15, Chap. 1], Ciarlet [17, Chaps. 3, 5] and the valuable recent contributions to quasi-static and dynamic elastic, viscoelastic and viscoplastic frictional-contact problems, with or without regularized friction and normal compliance, by Cocu et al. [18], Kuttler [19], Andersson [20], Andersson and Klarbring [21], Han and Sofonea [11, Parts 2–4], Shillor et al. [12, Sects. 2, 3], are essentially used. Here, an extension of the considered problem, material nonlinearities, functional peculiarities and approximation requirements has been presented. For example, one such extension of the variational treatment of frictional-contact problems in elasticity for rigid punch-indentation and extrusion problems, using large-deformation elastoplasticity, is given in [10, Chap. 12, Par. 5]. As a result, an incremental (updated Lagrangian) variational formulation is derived, to which existing methods are directly applicable. At high temperatures and loading conditions, characterizing continuous, hot metal-forming processes, however, the elastic deformations are negligible with respect to plastic ones. In such cases, the flow theory of plasticity with rigid–plastic, temperature, strain and strain-rate-dependent material model, adequately describes the material behaviour [1], [2, pp. 1–43], [3]. In [4]–[6] steady-state and nonsteady rolling problems with nonlocal friction, for rigid–plastic, incompressible, strain-rate and strain-dependent materials are considered. Variational inequality formulations are derived and existence and uniqueness results are obtained. Here, an extension of the variational and numerical study of the steady-state case is presented. The obtained results include: Lemma 3.1, establishing coercivity of the nonlinear functional (3.19); Theorem 4.1, giving an algorithmically oriented proof for existence and uniqueness of the solution of the penalty variational problem; Theorems 4.2 and 4.3, showing convergence of the solutions of the penalty and regularised penalty problem to the solutions of the primal and penalty one correspondingly, when the appropriately combined penalty and regularizations parameters tend to zero, and thus proving, as a consequence, existence and uniqueness of the solutions to these problems; Theorem 5.1, establishing an a priori finite-element error estimate of optimal order; Algorithm 1, an iterative computational scheme, combining finite-element and secant modulus methods and its verification for solving steady-state rolling problems. It has finally become clear that there are many fruitful topics for future research, related to metal-forming processes, some of which are: elaboration of the interface contact and friction models; variational formulation and analysis of new frictional-contact problems with coupled effects in the flow theory of plasticity with internal state variables; analysis and application of other approximation methods and computational algorithms.

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